Analytic Combinatorics Exercise Sheet 2

Exercises for the session on 3/4/2017

Problem 2.1

Let $y(z)$ satisfy $y = z\phi(y)$. Show that

$$zy' = \frac{y}{1 - z\phi'(y)}$$

and hence that for any function $a(y)$ we have

$$[z^n]\frac{ya(y)}{1 - z\phi'(y)} = [u^{n-1}]a(u)(\phi(u))^n \tag{1}$$

(you may use the formula $[z^n]H(y(z)) = \frac{1}{n}[u^{n-1}]H'(u) \cdot (\phi(u))^n$ for a function H).

Problem 2.2

Recall that the class of mappings from $\{1, 2, \ldots, n\}$ to itself has exponential generating function

$$F(z) = \frac{1}{1 - T(z)},$$

where $T(z) = ze^{T(z)}$. Use (1) to show

$$n![z^n]F(z) = n^n.$$

Problem 2.3

Recall that the class of mappings from $\{1, 2, \ldots, n\}$ to itself with no fixed points has exponential generating function

$$H(z) = \frac{e^{-T(z)}}{1 - T(z)},$$

where $T(z) = ze^{T(z)}$. Show

$$n![z^n]H(z) = (n-1)^n.$$

Problem 2.4

Recall that the exponential bivariate generating function of permutations counted according to cycles is $p(z, u) = (1 - z)^{-u}$. Show that the second moment of the number of cycles in a random permutation of size n satisfies

$$\mathbb{E}_{\mathcal{P}_n}(\chi(\chi - 1)) = \left(\sum_{k=1}^{n} \frac{1}{k}\right)^2 - \sum_{k=1}^{n} \frac{1}{k^2}$$

(you may use the binomial theorem for negative exponents $(1-z)^{-u} = \sum_{n\geq 0} \binom{u+n-1}{n} z^n$).

Problem 2.5

Let $A(z) = \sum_{n\geq 0} \frac{A_n}{n!} z^n$ denote the exponential generating function for the sequence defined by $A_0 = 1$ and $A_{n+1} = A_n + n$ for $n \geq 0$. Find $A'(z)$ in terms of $A(z)$, and hence find $A(z)$.

Problem 2.6

Find the exponential generating function for the number of surjective mappings from $\{1, 2, \ldots, n\}$ onto $\{1, 2, \ldots, k\}$ (for fixed k).